Journal of Pure and Applied Algebra 213 (2009) 194–202

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Qregularity and an extension of the Evans–Griffiths criterion to vector bundles on quadrics

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ARTICLE INFO

Article history:
Received 24 January 2008
Received in revised form 21 April 2008
Available online 3 July 2008
Communicated by A.V. Geramita

MSC:
14F05
14J60

ABSTRACT

Here we define the concept of Qregularity for coherent sheaves on a smooth quadric hypersurface $\mathcal{Q}_n \subset \mathbb{P}^{n+1}$. In this setting we prove analogs of some classical properties. We compare the Qregularity of coherent sheaves on $\mathcal{Q}_n$ with the Castelnuovo–Mumford regularity of their extension by zero in $\mathbb{P}^{n+1}$. We also classify the coherent sheaves with Qregularity $-\infty$. We use our notion of Qregularity in order to prove an extension of the Evans–Griffiths criterion to vector bundles on quadrics. In particular, we get a new and simple proof of Knörrer's characterization of ACM bundles.

1. Introduction

In Chapter 14 of [14] Mumford introduced the concept of regularity for a coherent sheaf on a projective space $\mathbb{P}^n$. Since then, Castelnuovo–Mumford regularity has become a fundamental invariant and was investigated by several mathematicians. Chipalkatti generalized this notion to coherent sheaves on Grassmannians [2] and Hoffman and Wang to coherent sheaves on multiprojective spaces [8]. Costa and Miró-Roig gave a definition of regularity for coherent sheaves on n-dimensional smooth projective varieties with an n-block collection [3–5].

The aim of this note is to introduce a very simple and natural concept of regularity (the Qregularity) on a smooth quadric hypersurface $\mathcal{Q}_n \subset \mathbb{P}^{n+1}$.

If we consider the following geometric collection on $\mathbb{P}^n$:

$$(\mathcal{E}_0, \dots, \mathcal{E}_n) = (\mathcal{O}_{\mathbb{P}^n}(-n), \mathcal{O}_{\mathbb{P}^n}(-n+1), \dots, \mathcal{O}_{\mathbb{P}^n}),$$

we obtain that a coherent sheaf F on $\mathbb{P}^n$ is said to be m-regular according to Castelnuovo–Mumford if

$$H^i(F(m) \otimes \mathcal{E}_{n-i}) = 0$$

for all $i = 1, \dots, n$. On the smooth quadric hypersurface $\mathcal{Q}_n$ we will use the n-block collection

$$(\mathcal{G}_0, \dots, \mathcal{G}_n) = (\mathcal{G}_0, \mathcal{O}_{\mathcal{Q}_n}(-n+1), \dots, \mathcal{O}_{\mathcal{Q}_n}),$$

where $\mathcal{G}_i = \mathcal{O}_{Q_n}(-n+i)$ for $1 \le i \le n$, $\mathcal{G}_0 = \Sigma(-n)$ if n is odd, and $\mathcal{G}_0 = (\Sigma_1(-n), \Sigma_2(-n))$ if n is even, where Σ_* are the spinor bundles (see Section 2 for their definition). We say that a coherent sheaf F on $\mathcal{Q}_n$ is m-Qregular if

$$H^i(F(m) \otimes \mathcal{G}_{n-i}) = 0$$

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doi:10.1016/j.jpaa.2008.06.002

for all $i = 1, \dots, n$. The interesting fact is that on $\mathcal{Q}_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ our definition of m-Qregularity coincides with the definition of (m, m)-regularity on $\mathbb{P}^1 \times \mathbb{P}^1$ given by Hoffman and Wang (see Remark 2.2). So we quote their results on $\mathbb{P}^1 \times \mathbb{P}^1$ as the starting step in order to prove on $\mathcal{Q}_n$ analogs of classical properties on $\mathbb{P}^{n+1}$ using induction on n as done by Mumford in the classical case $\mathbb{P}^n$. Next, we give some equivalent conditions of Qregularity. We compare the Qregularity of coherent sheaves on $\mathcal{Q}_n \subset \mathbb{P}^{n+1}$ with the Castelnuovo–Mumford regularity of their extension by zero in $\mathbb{P}^{n+1}$. We also classify coherent sheaves with Qregularity $-\infty$ as those with finite support. We also compare our definition of Qregularity with the one in [2] on $\mathcal{Q}_4 \cong G(2, 4)$ (Remark 3.8).

The second aim of this paper is to apply our notion of Qregularity in order to investigate under what circumstances a vector bundle can be decomposed into a direct sum of line bundles. A well-known result of Horrocks (see [9] or [15], p. 39) characterizes the vector bundles without intermediate cohomology on a projective space as direct sum of line bundles. This criterion fails on more general varieties. There exist non-split vector bundles without intermediate cohomology (Remark 4.5). These bundles are called ACM bundles. On $\mathbb{P}^n$, Evans and Griffiths (see [6]) have improved Horrocks' criterion:

Theorem 1.1 (*Evans–Griffiths*). *A rank r vector bundle E on $\mathbb{P}^n$, $(n \ge r > 0)$, splits if and only if $H^i(E(k)) = 0$ for all $i = 1, \dots, r-1$, and all $k \in \mathbb{Z}$.*

Knörrer classified all ACM bundles on a smooth quadric hypersurface $\mathcal{Q}_n$ as direct sums of line bundles and spinor bundles (up to a twist) [10]. Ottaviani generalized Horrocks criterion to quadrics and Grassmannians by giving cohomological splitting conditions for vector bundles [17].

Our main result is an extension of Evans–Griffiths criterion to vector bundles on quadrics. For any coherent sheaf A on $\mathcal{Q}_n$ and any integer $i \ge 0$, set $H^i_*(A) := \oplus_{t\in\mathbb{Z}} H^i(A(t))$. We improve Knörrer's theorem in the following way:

Theorem 1.2. *Let E be a rank r vector bundle on $\mathcal{Q}_n$, $n \ge 2$. Then the following conditions are equivalent:*

(a) *$H^i_*(E) = 0$ for every $i = 1, \dots, \min\{r-1, n-2\}$, and $H^{n-1}_*(E) = 0$.*
(b) *E is a direct sum of line bundles and twists of spinor bundles.*

In particular, we get a new and simpler proof of Knörrer's characterization of ACM bundles. Then we specialize to the case $r = 2$. We prove that if a Qregular rank 2 bundle E satisfies $H^1(E(-2)) = H^1(E(c_1(E))) = 0$, then it is a direct sum of line bundles and twists of spinor bundles. In particular, if $n > 4$, then $E \cong \mathcal{O} \oplus \mathcal{O}(c_1(E))$.

We work over an algebraically closed field $\mathbb{K}$ with characteristic zero. We only need the characteristic zero assumption to prove Theorem 1.2 and Proposition 4.6 because in their proofs we will use a Le Potier vanishing theorem.

We are grateful to E. Arrondo for showing us the connection between the notion of Qregularity and the splitting criteria for vector bundles.

2. *m*-Qregular coherent sheaves: Definition and properties

Let $\mathcal{Q}_n \subset \mathbb{P}^{n+1}$ be a smooth quadric hypersurface. We briefly recall Ottaviani's construction of the spinor bundles [16]. Set $k := \lfloor n/2 \rfloor$. For all intergers $m > r > 0$, let $G(r, m)$ denote the Grassmannian of all $(m-r)$-dimensional linear subspaces of $\mathbb{K}^m$. Let $U_{r,m}$ be the universal rank $(m-r)$ subbundle of $G(r, m)$. We first assume that n is odd. Ottaviani used the geometry of the variety of all k-dimensional linear subspaces of $\mathcal{Q}_n$ to construct a morphism $s_n : \mathcal{Q}_n \to G(2^k, 2^{k+1})$. Set $\Sigma_n(-1) := s_n^*(U_{2^k,2^{k+1}})$. Now assume that n is even. In this case we have two morphisms $s_{1,n} : Q_n \to G(2^{k-1}, 2^k)$ and $s_{2,n} : \mathcal{Q}_n \to G(2^{k-1}, 2^k)$. Set $\Sigma_{1,n}(-1) := s_{1,n}^*(U_{2^{k-1},2^k})$ and $\Sigma_{2,n}(-1) := s_{2,n}^*(U_{2^{k-1},2^k})$. If n is odd and we see $\mathcal{Q}_{n-1}$ as a smooth hyperplane section of $\mathcal{Q}_n$, then

$$\Sigma_n|Q_{n-1} \cong \Sigma_{1,n-1} \oplus \Sigma_{2,n-1}.$$

If n is even and we see $\mathcal{Q}_{n-1}$ as a smooth hyperplane section of $\mathcal{Q}_n$, then

$$\Sigma_{1,n}|\mathcal{Q}_{n-1} \cong \Sigma_{2,n}|\mathcal{Q}_{n-1} \cong \Sigma_{n-1}.$$

Since here we fix the integer n, we write Σ (resp. Σ_1 and Σ_2) instead of Σ_n (resp. $\Sigma_{1,n}$ and $\Sigma_{2,n}$) if n is odd (resp. even).

We use the unified notation Σ_* meaning that for even n both the spinor bundles Σ_1 and Σ_2 are considered, while $\Sigma_* = \Sigma$ if n is odd. We follow the notation of [3], i.e. the spinor bundles are twisted by 1 with respect to those of [16] ($\Sigma_* = \mathcal{S}_*(1)$).

Definition 2.1. A coherent sheaf F on $\mathcal{Q}_n$ $(n \ge 2)$ is said to be m-Qregular if $H^i(F(m-i)) = 0$ for $i = 1, \dots, n-1$, and $H^n(F(m) \otimes \Sigma_*(-n)) = 0$.

We will say Qregular instead of 0-Qregular.

Remark 2.2. A coherent sheaf F on $\mathcal{Q}_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ is m-Qregular if and only if $H^1(F(m-1, m-1)) = 0$ and $H^2(F(m, m) \otimes \Sigma_*(-2, -2)) = 0$. Since $\Sigma_1 \cong \mathcal{O}(1, 0)$ and $\Sigma_2 \cong \mathcal{O}(0, 1)$ if $n = 2$, our conditions become $H^1(F(m-1, m-1)) = 0$, $H^2(F(m-1, m-2)) = 0$, and $H^2(F(m-2, m-1)) = 0$. So the definition of m-Qregularity coincides with the definition of (m, m)-regularity on $\mathbb{P}^1 \times \mathbb{P}^1$ by Hoffman and Wang (see [8]).

Proposition 2.3. *Let F be an m-Qregular coherent sheaf on $\mathcal{Q}_n$ ($n \geq 2$).*

(a) *F is k-Qregular for all $k \geq m$.*
(b) *The natural morphism*

$$H^0(F(k-1)) \otimes H^0(\mathcal{O}(1)) \to H^0(F(k))$$

is surjective for all $k > m$.

Proof. We use induction on n. If $n = 2$, part (a) comes from [8], Proposition 2.7, and part (b) from [8], Proposition 2.8. Now assume $n = 3$. Let F be an m-Qregular coherent sheaf on $\mathcal{Q}_3$. There are finitely many closed subvarieties $T_i \subsetneq \mathcal{Q}_3$, $i \in S$, S a finite set, with the following property ([18], p. 26 and Theorem 1.11). For any $P \in Q_3$ set $S_P := \{i \in S : P \in T_i\}$. These subvarieties T_i, $i \in S$, support the points of $\mathcal{Q}_3$ on which the depth of F is generically constant and at most 2. Fix any effective Cartier divisor $D \subset \mathcal{Q}_3$. Then the natural map $F(-D) \to F$ is injective at P if and only $\dim(T_i \cap D) < \dim(T_i)$ for all $i \in S_P$ ([18], Theorem 1.14). Hence by Bertini's theorem ([11], part (b) of Th. 6.3) we may find a sufficiently general smooth hyperplane section $\mathcal{Q}_2$ of $\mathcal{Q}_3$ such that the following sequence

$$0 \to F(k-1) \to F(k) \to F_{|\mathcal{Q}_2}(k,k) \to 0 \tag{1}$$

is exact for all integers k. The long cohomology exact sequence of (1) gives the following exact sequence of vector spaces:

$$0 = H^1(F(m-1)) \to H^1(F_{|\mathcal{Q}_2}(m-1, m-1)) \to H^2(F(m-2)) = 0.$$

Thus $H^1(F_{|\mathcal{Q}_2}(m-1, m-1)) = 0$.

Look at the exact sequence on $\mathcal{Q}_3$:

$$0 \to F(m-2) \otimes \Sigma(-1) \to F(m-2)^4 \to F(m-2) \otimes \Sigma \to 0.$$

Since $H^2(F(m-2)) = 0$ and $H^3(F(m) \otimes \Sigma(-3)) = 0$, we also have $H^2(F(m) \otimes \Sigma(-2)) = 0$.

The m-Qregularity of F implies $H^2(F(m) \otimes \Sigma(-2)) = H^3(F(m) \otimes \Sigma(-3)) = 0$. Thus if we tensorize by $F(m)$ the exact sequence

$$0 \to \Sigma(-3) \to \Sigma(-2) \to \Sigma_{|\mathcal{Q}_2}(-2,-2) \to 0,$$

then we get $H^2(F_{|\mathcal{Q}_2}(m) \otimes \Sigma_1(-2,-2)) = H^2(F_{|\mathcal{Q}_2}(m) \otimes \Sigma_2(-2,-2)) = 0$.

So if F is m-Qregular on $\mathcal{Q}_3$, then $F_{|\mathcal{Q}_2}$ is an m-Qregular sheaf on $\mathcal{Q}_2$. Thus parts (a) and (b) are true for $F_{|\mathcal{Q}_2}$.

From the exact sequence on $\mathcal{Q}_2$:

$$0 \to F_{|\mathcal{Q}_2}(m-1, m-1) \otimes \Sigma_1(-1,-1) \to F_{|\mathcal{Q}_2}(m-1, m-1)^2 \to F_{|\mathcal{Q}_2}(m-1, m-1) \otimes \Sigma_2 \to 0,$$

we get the exact sequence

$$\begin{aligned} H^2(F_{|\mathcal{Q}_2}(m,m) \otimes \Sigma_1(-2,-2)) &\to H^2(F_{|\mathcal{Q}_2}(m-1, m-1))^2 \\ &\to H^2(F_{|\mathcal{Q}_2}(m+1, m+1) \otimes \Sigma_2(-2,-2)). \end{aligned} \tag{2}$$

Part (a) applied to $F_{|\mathcal{Q}_2}$ shows that the last vector space in (2) is 0. Since $F_{|\mathcal{Q}_2}$ is m-Qregular, the first vector space in (2) is 0. Therefore $H^2(F_{|\mathcal{Q}_2}(m-1, m-1)) = 0$.

Let us consider now the exact sequence

$$H^i(F(m-i)) \to H^i(F(m+1-i)) \to H^i(F_{|\mathcal{Q}_2}(m+1-i, m+1-i)). \tag{3}$$

By part (a) applied to $F_{|\mathcal{Q}_2}$ if $i = 1$, or by the above argument if $i = 2$, the last vector space in (3) is 0. The m-Qregularity of F gives that the first vector space in (3) vanishes if $i = 1, 2$. Thus $H^1(F(m+1-1)) = H^2(F(m+1-2)) = 0$. The exact sequence

$$H^3(F(m) \otimes \Sigma(-3)) \to H^3(F(m+1) \otimes \Sigma(-3)) \to 0$$

and the m-Qregularity of F give $H^3(F(m+1) \otimes \Sigma(-3)) = 0$.

Thus F is $(m+1)$-Qregular. Continuing in this way we prove part (a) for F.

To get part (b) we borrow the proof in [14], p. 100. Look at the case $x = 3$ of the following commutative diagram:

$$\begin{array}{ccc} H^0(F(k-1)) \otimes H^0(\mathcal{O}_{\mathcal{Q}_x}(1)) & \xrightarrow{\sigma} & H^0(F_{|\mathcal{Q}_{x-1}}(k-1, k-1)) \otimes H^0(\mathcal{O}_{\mathcal{Q}_{x-1}}(1,1)) \\ \downarrow \mu & & \downarrow \tau \\ H^0(F(k)) & \xrightarrow{\nu} & H^0(F_{|\mathcal{Q}_{x-1}}(k,k)). \end{array} \tag{4}$$

Since $H^1(F(k-2)) = 0$ if $k > m$, σ is surjective for all $k > m$.

Now we prove the surjectivity of τ for all $k > m$. From [8], Proposition 2.8, we know that $H^0(F_{|\mathcal{Q}_2}(k,k))$ is spanned by

$$H^0(F_{|\mathcal{Q}_2}(k-1, k)) \otimes H^0(\mathcal{O}_{|\mathcal{Q}_2}(1,0))$$

and also by

$$H^0(F_{|\mathcal{Q}_2}(k,k-1)) \otimes H^0(\mathcal{O}_{|\mathcal{Q}_2}(0,1)).$$

Thus both maps

$$H^0(F_{|\mathcal{Q}_2}(k-1,k-1)) \otimes H^0(\mathcal{O}_{|\mathcal{Q}_2}(1,0)) \otimes H^0(\mathcal{O}_{|\mathcal{Q}_2}(0,1)) \to H^0(F_{|\mathcal{Q}_2}(k-1,k)) \otimes H^0(\mathcal{O}_{|\mathcal{Q}_2}(1,0))$$

and

$$H^0(F_{|\mathcal{Q}_2}(k-1,k)) \otimes H^0(\mathcal{O}_{|\mathcal{Q}_2}(1,0)) \to H^0(F_{|\mathcal{Q}_2}(k,k))$$

are surjective. Hence their composition is surjective. The surjection

$$H^0(\mathcal{O}_{|\mathcal{Q}_2}(1,0)) \otimes H^0(\mathcal{O}_{|\mathcal{Q}_2}(0,1)) \to H^0(\mathcal{O}_{|\mathcal{Q}_2}(1,1))$$

shows that the map

$$H^0(F_{|\mathcal{Q}_2}(k-1,k-1)) \otimes H^0(\mathcal{O}_{|\mathcal{Q}_2}(1,1)) \xrightarrow{\tau} H^0(F_{|\mathcal{Q}_2}(k,k)),$$

is surjective. Let $\nu : H^0(F(k)) \to H^0(F|_{\mathcal{Q}_2}(k,k))$ denote the restriction map. Let $z \in H^0(\mathcal{O}(1))$ be an equation of $\mathcal{Q}_2$ in $\mathcal{Q}_3$. Since both σ and τ are surjective, the commutative diagram (4) gives the surjectivity of $\nu \circ \mu$. Every element of $\mathrm{Ker}(\nu)$ is of the form $z\beta$ for some $\beta \in H^0(F(k-1))$. Thus $\mathrm{Ker}(\nu) \subseteq \mathrm{Im}(\mu)$. Thus μ is surjective.

We assume parts (a) and (b) on $\mathcal{Q}_{2n-1}$. We need to prove them on $\mathcal{Q}_{2n}$.

Let F be an m-Qregular coherent sheaf on $\mathcal{Q}_{2n}$. Fix a general hyperplane section $\mathcal{Q}_{2n-1}$ of $\mathcal{Q}_{2n}$. For any integer k the following sequence on $\mathcal{Q}_{2n}$ is exact:

$$0 \to F(k-1) \to F(k) \to F_{|Q_{2n-1}}(k) \to 0.$$

(use [18] and Bertini's theorem as in the case $n = 3$). Thus we get the exact sequence on $\mathcal{Q}_{2n}$:

$$H^i(F(m-i)) \to H^i(F_{|Q_{2n-1}}(m-i)) \to H^{i+1}(F(m-i-1)).$$

Hence $H^i(F_{|\mathcal{Q}_{2n-1}}(m-i)) = 0$ for $i = 1, \dots, 2n-2$.

Let us consider now the exact sequences on $\mathcal{Q}_{2n}$ ([16], Theorem 2.8):

$$0 \to F(m-2n+1) \otimes \Sigma_1(-1) \to F(m-2n+1)^{2^n} \to F(m-2n+1) \otimes \Sigma_2 \to 0 \tag{5}$$

$$0 \to F(m-2n+1) \otimes \Sigma_2(-1) \to F(m-2n+1)^{2^n} \to F(m-2n+1) \otimes \Sigma_1 \to 0. \tag{6}$$

Since $H^{2n-1}(F(m-2n+1)) = 0$ and $H^{2n}(F(m) \otimes \Sigma_*(-2n)) = 0$, these exact sequences give $H^{2n-1}(F(m) \otimes \Sigma_*(-2n+1)) = 0$.

If we tensorize by $F(m)$ the exact sequence

$$0 \to \Sigma_1(-2n) \to \Sigma_1(-2n+1) \to \Sigma_{1|\mathcal{Q}_{2n-1}}(-2n+1) \to 0$$

and use again that $\mathcal{Q}_{2n-1}$ is general, then we get the exact sequence

$$0 \to F(m) \otimes \Sigma_1(-2n) \to F(m) \otimes \Sigma_1(-2n+1) \to F_{|Q_{2n-1}}(m) \otimes \Sigma_1(-2n+1) \to 0.$$

So we obtain the exact sequence

$$H^{2n-1}(F(m) \otimes \Sigma_1(-2n+1)) \to H^{2n-1}(F_{|\mathcal{Q}_{2n-1}}(m) \otimes \Sigma_1(-2n+1)) \to H^{2n}(F(m) \otimes \Sigma_1(-2n)).$$

Thus $H^{2n-1}(F_{|\mathcal{Q}_{2n-1}}(m) \otimes \Sigma_1(-2n+1)) = 0$.

So if F is an m-Qregular sheaf on $\mathcal{Q}_{2n}$, then $F_{|\mathcal{Q}_{2n-1}}$ is an m-Qregular sheaf on $\mathcal{Q}_{2n-1}$. Hence the inductive assumption gives parts (a) and (b) for $F_{|\mathcal{Q}_{2n-1}}$.

From the exact sequence on $\mathcal{Q}_{2n-1}$:

$$0 \to F_{|\mathcal{Q}_{2n-1}}(m) \otimes \Sigma(-2n+1) \to F_{|\mathcal{Q}_{2n-1}}(m-2n+2)^{2^n} \to F_{|\mathcal{Q}_{2n-1}}(m+1) \otimes \Sigma(-2n+1) \to 0,$$

we get the exact sequence

$$\begin{aligned} &H^{2n-1}(F_{|\mathcal{Q}_{2n-1}}(m) \otimes \Sigma(-2n+1)) \to H^{2n-1}(F_{|\mathcal{Q}_{2n-1}}(m-2n+2))^{2^n} \\ &\quad \to H^{2n-1}(F_{|\mathcal{Q}_{2n-1}}(m+1) \otimes \Sigma(-2n+1)). \end{aligned} \tag{7}$$

Part (a) applied to $F_{|\mathcal{Q}_{2n-1}}$ gives that the last vector space in (7) is 0. The m-Qregularity of F gives that the first vector space in (7) is 0. Thus $H^{2n-1}(F_{|\mathcal{Q}_{2n-1}}(m-2n+2)) = 0$. Now we use the following exact sequence:

$$H^i(F(m-i)) \to H^i(F(m+1-i)) \to H^i(F_{|\mathcal{Q}_{2n-1}}(m+1-i)). \tag{8}$$

By part (a) for $F_{|\mathcal{Q}_{2n-1}}$ if $i = 1, \dots 2n-2$, or by the above argument if $i = 2n-1$, the last vector space in (8) is 0. The m-Qregularity of F gives that the first vector space in (8) is 0 if $i = 1, \dots 2n-1$. Therefore $H^i(F(m+1-i)) = 0$ for $i = 1, \dots 2n-1$. From the exact sequence

$$H^{2n}(F(m) \otimes \Sigma(-2n)) \to H^{2n}(F(m+1) \otimes \Sigma(-2n)) \to 0$$

we also see that $H^{2n}(F(m+1) \otimes \Sigma(-2n)) = 0$.

We get that F is $(m+1)$-Qregular. By iterating this argument, we prove part (a) for F.

Now we prove part (b). Consider the case $x = 2n$ of the commutative diagram (4). If $k > m$, the map σ is surjective, because $H^1(F(k-2)) = 0$. If $k > m$, then τ is surjective by part (b) for $F_{|\mathcal{Q}_{2n-1}}$. Since both σ and τ are surjective, the commutative diagram (4) gives the surjectivity of μ as we have explained for Q_3.

In a very similar way, we can prove parts (a) and (b) on $\mathcal{Q}_{2n+1}$, assuming their truth on $\mathcal{Q}_{2n}$. □

Here there are some equivalent definitions of m-Qregular coherent sheaves:

Proposition 2.4. *Let F be a coherent sheaf on $\mathcal{Q}_n$. The following conditions are equivalent:*

(a) *F is m-Qregular.*
(b) *$H^i(F(m-i)) = 0$ for $i = 1, \dots, n-1$, $H^{n-1}(F(m) \otimes \Sigma_*(-n+1)) = 0$, and $H^n(F(m-n+1)) = 0$.*

Proof. If n is odd, then there is an exact sequence

$$0 \to F(k) \otimes \Sigma(-1) \to F(k)^{2^{((n+1)/2)}} \to F(k) \otimes \Sigma \to 0 \tag{9}$$

([16]; Theorem 2.8). (a) $\Rightarrow$ (b). Let F be m-Qregular. Hence F is $(m+1)$-Qregular (Proposition 2.3). Thus $H^n(F(m+1) \otimes \Sigma_*(-n-1)) = H^n(F(m+1) \otimes \Sigma_*(-n)) = 0$ and $H^n(F(m+1) \otimes \Sigma_*(-n-1)) = H^{n-1}(F(m+1-n)) = 0$. Hence either (9) (if n is even) or (5) (if n is odd) give $H^n(F(m+1-n)) = 0$ and $H^{n-1}(F(m+1) \otimes \Sigma_*(-n)) = 0$. So (b) is true. $(b) \Rightarrow (a)$. Let F be a coherent sheaf which satisfies (b). Since $H^{n-1}(F(m) \otimes \Sigma_*(-n+1)) = H^n(F(m-n+1)) = 0$, the exact sequences (9) or (5) give $H^n(F(m) \otimes \Sigma_*(-n)) = 0$. Hence F is m-regular. □

Now we show that any Qregular coherent sheaf is globally generated:

Proposition 2.5. *Any Qregular coherent sheaf F on $\mathcal{Q}_n$ is globally generated.*

Proof. We need to prove that the evaluation map

$$\varphi : H^0(F) \otimes \mathcal{O}_{Q_n} \to F$$

is surjective. This is equivalent to proving that its tensor product with id_Σ is surjective, because this would imply that

$$\varphi \otimes id_\Sigma \otimes id_{\Sigma^\vee} : H^0(F) \otimes \Sigma \otimes \Sigma^\vee \to F \otimes \Sigma \otimes \Sigma^\vee$$

is surjective, and $\Sigma \otimes \Sigma^\vee$ is faithfully flat. The following diagram is commutative :

$$\begin{array}{ccc} H^0(F) \otimes H^0(\Sigma) \otimes \mathcal{O}_{Q_n} & \xrightarrow{\eta} & H^0(F \otimes \Sigma) \otimes \mathcal{O}_{Q_n} \\ \downarrow & & \downarrow \psi \\ H^0(F) \otimes \Sigma & \xrightarrow{\varphi \otimes id_\Sigma} & F \otimes \Sigma. \end{array}$$

Thus it is enough to prove that η and ψ are surjective. Proposition 2.4 gives $H^1(F \otimes \Sigma_*(-1)) = 0$. Hence η is surjective, and more generally $F \otimes \Sigma$ is 0-regular in the sense of Castelnuovo–Mumford. Hence $F \otimes \Sigma$ is globally generated. Thus ψ is surjective. □

3. Qregularity on $\mathcal{Q}_n$

Definition 3.1. Let F be a coherent sheaf on $\mathcal{Q}_n$. We define the Qregularity of F, Qreg(F), as the least integer m such that F is m-Qregular. We set Qreg$(F) = -\infty$ if there is no such an integer.

Remark 3.2. Here we work on $\mathcal{Q}_n$, $n \geq 2$, and check that Qreg$(\mathcal{O})$ = Qreg$(\Sigma_*) = 0$. Indeed, $\mathcal{O}$ and Σ_* are ACM bundles. $H^n(\mathcal{O} \otimes \Sigma_*(-n)) \cong H^0(\Sigma^\vee) = H^0(\Sigma(-1)) = 0$ and $H^n(\Sigma_* \otimes \Sigma_*(-n)) \cong H^0(\Sigma_* \otimes \Sigma_*(-2)) = 0$. So $\mathcal{O}$ and Σ_* are 0-Qregular. Since $h^0(\mathcal{O}(-1)) = h^0(\Sigma_*(-1)) = 0$, Proposition 2.5 implies that $\mathcal{O}$ and Σ_* are not (-1)-Qregular.

Remark 3.3. Let

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

be an exact sequence of coherent sheaves on $\mathcal{Q}_n$. Then Qreg$(F_2) \leq \max\{$Qreg(F_1), Qreg$(F_3)\}$. Let F and G be coherent sheaves on $\mathcal{Q}_n$. Then Qreg$(F \oplus G) = \max\{$Qreg(F), Qreg$(G)\}$.

Let F be a coherent sheaf on $\mathcal{Q}_n$ $(n > 1)$, and let $i_*(F)$ be its extension by zero in the embedding $i : \mathcal{Q}_n \hookrightarrow \mathbb{P}^{n+1}$. Now we compare the Qregularity of F with the regularity in the sense of Castelnuovo–Mumford of $i_*(F)$. We recall the following definition.

Definition 3.4. A coherent sheaf F on $\mathcal{Q}_n$ is said to be m-regular in the sense of Castelnuovo–Mumford if $H^j(\mathbb{P}^{n+1}, i_*(F)(m-i)) = 0$ for all $j = 1 \dots n+1$.

Reg(F) is the least integer m such that F is m-regular. Set Reg$(F) := -\infty$ if there is no such an integer.

Proposition 3.5. *Let $i : \mathcal{Q}_n \hookrightarrow \mathbb{P}^{n+1}$ be a quadric hypersurface $(n > 1)$. Then*

$$\mathrm{Qreg}(F) \le \mathrm{Reg}(i_*(F)) \le \mathrm{Qreg}(F) + 1$$

for every coherent sheaf F on $\mathcal{Q}_n$.

Proof. We have to prove that if $i_*(F)$ is m-regular, then F is m-Qregular, and that if F is $(m-1)$-Qregular, then $i_*(F)$ is m-regular.

Note that $i_*(F)(t) = i_*(F(t))$ for every $t \in \mathbb{Z}$. Since the closed embedding $i : \mathcal{Q}_n \hookrightarrow \mathbb{P}^{n+1}$ is an affine morphism,

$$H^j(\mathbb{P}^{n+1}, i_*(F(t))) = H^j(\mathcal{Q}_n, F(t))$$

for all $(t, j) \in \mathbb{Z} \times \mathbb{N}$. Hence $i_*(F)$ is m-regular if and only if $H^j(\mathcal{Q}_n, F(m-j)) = 0$ for all $j = 1 \dots n$.

To prove that F is m-Qregular we only need to prove that $H^n(\mathcal{Q}_n, F(m) \otimes \Sigma_*(-n)) = 0$. Setting $k = m - n$ in the exact sequence (9) if n is odd or using the exact sequence (Remark 2.2 if $n = 2\widetilde{n}$ is even), we find that if $H^n(\mathcal{Q}_n, F(m-n)) = 0$, then $H^n(F(m) \otimes \Sigma_*(-n)) = 0$. Let F be $(m-1)$-Qregular. Since F is m-Qregular (Proposition 2.3), we only need to prove that $H^n(\mathcal{Q}_n, F(m-n)) = 0$. The same exact sequence shows that if $H^n(F(m-1) \otimes \Sigma_*(-n)) = H^n(F(m) \otimes \Sigma_*(-n)) = 0$, then $H^n(\mathcal{Q}_n, F(m-n)) = 0$. □

Remark 3.6. Now we show that Proposition 3.5 is optimal. Remark 3.2 gives $\mathrm{Qreg}(\mathcal{O}) = \mathrm{Qreg}(\Sigma_*) = 0$. Here we check that $\mathrm{Reg}(i_*(\mathcal{O})) = 1$ and $\mathrm{Reg}(i_*(\Sigma_*)) = 0$ for all $n > 2$. Indeed,

$$H^n(\mathbb{P}^{n+1}, i_*(\mathcal{O})(t-n)) = H^n(\mathcal{Q}_n, \mathcal{O}(t-n)) = 0$$

if and only if $t \ge 1$, and

$$H^n(\mathbb{P}^{n+1}, i_*(\Sigma_*(t-n))) = H^n(\mathcal{Q}_n, \Sigma_*(t-n)) = 0$$

if and only if $t \ge 0$.

Theorem 3.7. *Let F be a coherent sheaf on $\mathcal{Q}_n$ (n even). The following conditions are equivalent:*

1. $\mathrm{Qreg}(F) = -\infty$.
2. $\mathrm{Reg}(F) = -\infty$.
3. $\mathrm{Supp}(F)$ *is finite.*

Let F be a coherent sheaf on $\mathcal{Q}_n$ (n odd). Let us consider the geometric collection on $\mathcal{Q}_n$:

$$\sigma = (\mathcal{O}, \dots, \mathcal{O}(n-1), \Sigma(-n-1)).$$

The following conditions are equivalent:

1. $\mathrm{Qreg}(F) = -\infty$.
2. $\mathrm{Reg}(F) = -\infty$.
3. $\mathrm{Reg}_\sigma(F) = -\infty$.
4. $\mathrm{Supp}(F)$ *is finite.*

Proof. Let F be a coherent sheaf on $\mathcal{Q}_n$. By Proposition 3.5 $\mathrm{Qreg}(F) = -\infty$ if and only if $\mathrm{Reg}(F) = -\infty$. By [1], Theorem 1, $Reg(F) = -\infty$ if and only if $\mathrm{Supp}(F)$ is finite. Let n be an odd integer. By [3], Theorem 4.3, $Reg(F) = -\infty$ if and only if $\mathrm{Reg}_\sigma(F) = -\infty$. By [1], Theorem 1, $Reg_\sigma(F) = -\infty$ if and only if $\mathrm{Supp}(F)$ is finite. □

Remark 3.8. Here we compare our definition of Qregularity on $\mathcal{Q}_4 \cong G(2, 4)$ with the one introduced in [2] and with Castelnuovo–Mumford regularity. Let S (resp. Q) denote the universal rank 2 subbundle (resp. quotient bundle) on $G(2, 4)$. Note that $\{\Sigma_1, \Sigma_2\} = \{S(1), Q\}$. A coherent sheaf F on $G(2, 4)$ is said to be m-regular in the sense of [2] or m-$\mathbb{G}$-regular if

$$H^i(F(m) \otimes A) = 0$$

for every integer $i > 0$ and every vector bundle A in the following set

$$\Psi := \{\mathcal{O}, Q^\vee, \mathrm{Sym}^2 Q^\vee, \mathcal{O}(-1), Q^\vee(-1), \mathcal{O}(-2)\}$$

(see [2], bottom of page 451). The $\mathbb{G}$-regularity of F is the minimal integer m such that F is m-$\mathbb{G}$-regular. Let m (resp. m_1, resp. m_2) be the $\mathbb{G}$-regularity (resp. Qregularity, resp. Castelnuovo–Mumford regularity) of F. Hence $m_1 \le m_2 \le m_1 + 1$

(Proposition 3.5). The sheaf F has $\mathbb{G}$-regularity $-\infty$ if and only if it has finite support ([2], Remark 1.2.4), i.e. if and only if it has Qregularity $-\infty$ (Remark 4.1). Hence from now on we assume that m, m_1, and m_2 are finite integers. Chipalkatti's definition of $\mathbb{G}$-regularity is not invariant under the action of the order two automorphism

$$\sigma : \mathcal{Q}_4 \to \mathcal{Q}_4,$$

which exchanges Q and $S(1)$, i.e. which exchanges Σ_1 and Σ_2 (see [2], Remark 1.16). Indeed, $S^\vee$ has Qregularity 0 and $\mathbb{G}$-regularity 1, while Q has Qregularity and $\mathbb{G}$-regularity 0 ([2], Remark 1.16). The $\mathbb{G}$-regularity $\widetilde{m}$ of F with respect to the identification of $\mathcal{Q}_4$ with a dual Grassmannian is given using the vanishing of the cohomology groups $H^i(F(t) \otimes A)$ for all integers $t \geq \widetilde{m}$ and $i > 0$ and all vector bundles A in the following set

$$\widetilde{\Psi} := \{\mathcal{O}, S, \mathrm{Sym}^2 S, \mathcal{O}(-1), S(-1), \mathcal{O}(-2)\}.$$

Since $\mathcal{O}(-2) \in \Psi \cap \widetilde{\Psi}$, $m_2 - 2 \leq \min\{m, \widetilde{m}\}$. Recall that Q and $S(1)$ have regularity 0 in the sense of Castelnuovo–Mumford. Since the tensor product of two Castelnuovo–Mumford regular vector bundles is regular in the sense of Castelnuovo–Mumford and $Q^\vee \cong Q(-1)$, $\mathrm{Sym}^2 Q^\vee$ has Castelnuovo–Mumford index of regularity ≤ 2. Thus

$$h^i(F(t) \otimes \mathrm{Sym}^2 Q^\vee) = 0$$

for all $(i, t) \in \mathbb{Z}^2$ such that $i > 0$ and $t \geq m_2 + 2 - i$. Similarly,

$$h^i(F(t) \otimes Q^\vee) = 0$$

for all $(i, t) \in \mathbb{Z}^2$ such that $i > 0$ and $t \geq m_2 + 1 - i$. Thus $m \leq m_2 + 2$. Similarly, $\widetilde{m} \leq m_2 + 2$. We do not have a better comparison between m and m_1, except the one coming from the inequalities

$$m_2 - 2 \leq m \leq m_2 + 2$$

just proved and the inequalities $m_1 \leq m_2 \leq m_1 + 1$ (Proposition 3.5).

4. Evans–Griffiths criterion on quadrics

Remark 4.1. In the proofs of Theorem 1.2 and of Proposition 4.6 we are going to use the following weak form of Le Potier vanishing theorem, i.e. the case $a = 1$ of the first assertion of [12], Theorem 7.3.5. Let X be an n-dimensional smooth projective variety, and let E be a rank r ample vector bundle on X. Then $H^i(X, E \otimes \omega_X) = 0$ for all $i \geq r$. Note that this vanishing theorem says something only if $r \leq n$. If $r \geq n - 1$, then Theorem 1.2 was proved a long time ago [10,17].

Proof of Theorem 1.2. Since E is locally free, Theorem 3.7 shows that E has finite Qregularity. Hence, up to a twist, we may assume that E is Qregular, while $E(-1)$ is not. Since E is Qregular, it is globally generated (Proposition 2.5). Thus $E(1)$ is ample. Le Potier vanishing theorem (see Remark 4.1) gives $H^i(E(-n + l)) = 0$ for all $(l, i) \in \mathbb{Z}^2$ such that $l > 0$ and $r \leq i \leq n - 1$. In particular, $H^i(E(-1 - i)) = 0$ for $r \leq i \leq n - 2$. We assumed $H^i(E(-1 - i)) = 0$ for $i = 1, \ldots, \min\{n - 2, r - 1\}$, and $H^{n-1}(E(-1 - n + 1)) = 0$. Thus $E(-1)$ is not Qregular if and only if either $H^{n-1}(E(-1) \otimes \Sigma_*(-n + 1)) \neq 0$ or $H^n(E(-1 - n + 1)) \neq 0$. We first assume $H^n(E(-1 - n + 1)) \neq 0$, i.e. $H^0(E^\vee) \neq 0$ (Serre duality). Thus there is a non-zero map $f : E \to \mathcal{O}$. Since E is globally generated, there is a map $g : \mathcal{O} \to E$ such that $f \circ g \neq 0$. Hence $f \circ g$ is a non-zero multiple of the identity map $\mathcal{O} \to \mathcal{O}$. Hence $\mathcal{O}$ is a direct summand of E. Now we assume $H^{n-1}(E(-1) \otimes \Sigma_*(-n + 1)) \neq 0$ and $H^n(E(-1 - n + 1)) = 0$. We first handle the case n even. We consider the following exact sequence:

$$0 \to E(k) \otimes \Sigma_2(-1) \to E(k)^{2^{n/2}} \to E(k) \otimes \Sigma_1 \to 0.$$

Since $H^n(E(-1 - n + 1)) = H^{n-1}(E(-1 - n + 1)) = 0$, we see that

$$H^{n-1}(E(-1) \otimes \Sigma_1(-n + 1)) \cong H^n(E(-1) \otimes \Sigma_2(-n)).$$

Thus Serre duality gives $H^0(E^\vee(1) \otimes \Sigma_2^\vee) \neq 0$ and the existence of a non-zero map

$$\eta : E(-1) \to \Sigma_2^\vee.$$

On the other hand, since $H^j(E(-1 - j)) = 0$ for every $j = 1, \ldots, n - 1$, all the following maps

$$H^0(E \otimes \Sigma_2(-1)) \to H^1(E \otimes \Sigma_1(-2)) \to \cdots \to H^{n-2}(E \otimes \Sigma_2(-n + 1)) \to H^{n-1}(E \otimes \Sigma_1(-n))$$

are surjective. Thus $H^0(E(-1) \otimes \Sigma_2) \neq 0$, and there is a non-zero map

$$\beta : \Sigma_2^\vee \to E(-1).$$

Look at the following commutative diagram:

$$\begin{array}{ccc}
H^{n-1}(E(-1)\otimes\Sigma_1(-n+1))\otimes H^1(E^\vee(1)\otimes\Sigma_1^\vee(-1)) & \xrightarrow{\sigma} & H^n(\Sigma_1^\vee(-1)\otimes\Sigma_1(-n+1))\cong\mathbb{C}\\
\downarrow & & \downarrow\\
H^0(E(-1)\otimes\Sigma_2)\otimes H^1(E^\vee(1)\otimes\Sigma_1^\vee(-1)) & \xrightarrow{\mu} & H^1(\Sigma_1^\vee(-1)\otimes\Sigma_2)\cong\mathbb{C}\\
\downarrow & & \downarrow\\
H^0(E(-1)\otimes\Sigma_2)\otimes H^0(E^\vee(1)\otimes\Sigma_2^\vee) & \xrightarrow{\tau} & H^0(\Sigma_2^\vee\otimes\Sigma_2)\cong\mathbb{C}\\
\uparrow\cong & & \uparrow\cong\\
\mathrm{Hom}(\Sigma_2^\vee,E(-1))\otimes\mathrm{Hom}(E(-1),\Sigma_2^\vee) & \xrightarrow{\gamma} & \mathrm{Hom}(\Sigma_2^\vee,\Sigma_2^\vee)
\end{array}$$

The map σ comes from Serre duality and it is not zero. The right vertical maps are isomorphisms. The left vertical maps are surjective, and $\tau\neq 0$. Thus $\eta\circ\beta\neq 0$. Hence $\eta\circ\beta$ is a non-zero multiple of the identity. Hence $\Sigma_2^\vee$ is a direct summand of $E(-1)$. By [16], Theorem 2.8, we have

$$\Sigma_2^\vee\cong\begin{cases}\Sigma_2(-1) & \text{if } n\equiv 0 \pmod 4\\ \Sigma_1(-1) & \text{if } n\equiv 2 \pmod 4.\end{cases}$$

In the same way we can prove that $\Sigma_1^\vee$ is a direct summand of $E(-1)$ if $H^{n-1}(E(-1)\otimes\Sigma_2(-n+1))\neq 0$. If n is odd we see in the same way that the assumption $H^{n-1}(E(-1)\otimes\Sigma_*(-n+1))\neq 0$ gives that $\Sigma^\vee$ is a direct summand of $E(-1)$. By iterating these arguments, we obtain that E is a direct sum of line bundles and twists of spinor bundles. □

Remark 4.2. Make all the assumptions of Theorem 1.2. Recall that $\mathrm{rank}(\Sigma_*)=2^{\lfloor(n-1)/2\rfloor}$. Hence E has no factor isomorphic to $\Sigma_*(t)$ if $r<2^{\lfloor(n-1)/2\rfloor}$. Hence E is a direct sum of line bundles if $r<2^{\lfloor(n-1)/2\rfloor}$.

As a Corollary of Theorem 1.2 we get the following two splitting criteria:

Corollary 4.3. *Let E be a rank r vector bundle on $\mathcal{Q}_n$, $n\geq 2$. Then the following conditions are equivalents:*

1. $H^i_*(E)=0$ *for all* $i=1,\ldots,\min\{r-1,n-2\}$, $H^{n-1}_*(E)=0$ *and* $H^{n-1}_*(E\otimes\Sigma_*)=0$,
2. *E is a direct sum of line bundles.*

Corollary 4.4 (*Knörrer*). *Let E be a rank r vector bundle on $\mathcal{Q}_n$ such that $H^i_*(E)=0$ for all $i=1,\ldots,n-1$. Then E is a direct sum of line bundles and twists of spinor bundles.*

Remark 4.5. The hypothesis $H^{n-1}_*(E)=0$ does not appear in the Evans–Griffiths criterion on $\mathbb{P}^n$. On $\mathcal{Q}_n$ it is necessary, because on $\mathcal{Q}_n$ there are many indecomposable bundles with $H^1_*(E)=\cdots=H^{n-2}_*(E)=0$ but $H^{n-1}_*(E)\neq 0$. On $\mathcal{Q}_4$ there is the rank 3 bundle P_4 arising from the following exact sequence (see [7] or [13]):

$$0\to\mathcal{O}\to\Sigma_1\oplus\Sigma_2\to P_4\to 0.$$

On $\mathcal{Q}_5$ there is the rank 3 bundle P_5 arising from the following exact sequence (see [13]):

$$0\to\mathcal{O}\to\Sigma\to P_5\to 0.$$

If $\mathrm{rank}(E)=2$, then $E^\vee\cong E(c_1(E))$. Hence the assumption $H^{n-1}_*(E)=0$ may be omitted if $\mathrm{rank}(E)=2$.

Proposition 4.6. *Let E be a rank 2 bundle on $\mathcal{Q}_n$ with* $\mathrm{Qreg}(E)=0$ *and $H^1(E(-2))=H^1(E(c_1))=0$, where $c_1:=c_1(E)$. Then E is a direct sum of line bundles and twists of spinor bundles. If $n>4$, then $E\cong\mathcal{O}\oplus\mathcal{O}(c_1)$.*

Proof. Since $\mathrm{rank}(E)=2$, $E^\vee\cong E(-c_1)$. Since E is Qregular, it is globally generated (Proposition 2.5). Thus $E(1)$ is ample. Thus Le Potier vanishing theorem (Remark 4.1) gives $H^i(E(-n+l))=0$ for every $l>0$ and $i=2,\ldots,n-1$. Hence $H^i(E(-1-i))=0$ for $i=2,\ldots,n-2$. By assumption $H^i(E(-1-i))=0$ for $i=1,\ldots,n-2$ and $H^{n-1}(E(-1-n+1))\cong H^1(E(c_1))=0$. Thus $E(-1)$ is not Qregular if and only if $H^{n-1}(E(-1)\otimes\Sigma_*(-n+1))\neq 0$ or $H^n(E(-1-n+1))\neq 0$. Arguing as in the proof of Theorem 1.2 we show that either $\mathcal{O}$ or $\Sigma_*^\vee$ is a direct summand of E.

If $n>4$, then $\Sigma_*^\vee$ is not a direct factor of E, because $\mathrm{rank}(\Sigma_*^\vee)\geq 3$ for all $n>4$. □

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